RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 48, NUMBER 2 AUGUST 1993

Random walker and the telegrapher's equation: A paradigm of a generalized hydrodynamics

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(Received 7 January 1993)

The telegrapher's equation (TE) is the continuum limit of a persisting random walker. We find that the TE reproduces the original spectrum almost exactly for all wavelengths—far beyond the validity of the expansion. This surprising property is used as a paradigm towards the derivation of a generalized hydrodynamics. Applications to other problems are explored.

PACS number(s): 51.10.+y, 05.40.+j, 05.50.+q, 05.60.+w

When the motion of a persisting random walker is taken to its continuum limit, the dynamics is then given by the telegrapher's equation (TE). In this Rapid Communication we first show that the spectrum of the TE reproduces almost exactly for all wavelengths the spectrum of the original, discrete process. We then explore the implications of this surprising spectral proximity with the intention of using it as a guiding paradigm in kinetic theory. The implications of this result may very well apply to other problems as well. In general one does not have the right to expect the ultraviolet limits of a continuum model and its microscopic antecedent to have much in common—hence the surprise. In the present problem *the validity of the results predicted by the TE greatly exceeds the validity of the derivation that begets it.*

To put things into perspective, we remind the reader that the spectral behavior of the time-discrete or time-continuous, completely uncorrelated walker and the continuum limit of this process, the Fokker-Planck equation, are completely different in the ultraviolet regime. In fact, this high-k discrepancy of the Fokker-Planck equation is considered to be one of its major deficiencies [1].

Technically, the spectral proximity of the TE to the persisting random walker can be traced to the effects of memory built into the original process and to the fact that the expansion preserves V, the characteristic speed of the walker. This speed is the basic invariant of the original process. That our expansion preserves such invariants is crucial. It allows us to circumvent the limitations usually imposed by a long-wavelength expansion. The process described by the resulting equations is endowed with some important short-wavelength features of the original process, which otherwise would be lost. This property was probably noted before but does not seem to be appreciated or, what is more important, explored.

Extrapolating beyond the technical aspects of the present problem, one recognizes the general importance of a process with memory and an expansion that does not destroy the basic invariants of the process. In such an expansion, space and time must play an equally important role. The hyperbolic nature of the resulting, continuum equations is the macroscopic manifestation of such an expansion. We expect that these equations will remain, in the spectral sense, uniformly close to the original process.

The present problem provides us with a new outlook on some of the classical results, the most notable of which are the Navier-Stokes equations—the NSE's. The NSE's are the gas-dynamic analogy of the Fokker-Planck equation—the low-k universe of the uncorrelated random walker. In the Chapman-Enskog expansion of the Boltzmann equation, which begets the NSE's, time and space are not kept on equal footing. Time enters as a passive variable that is eliminated in favor of its spatial counterpart, rendering the process completely local timewise. The elimination is achieved at the expense of an ever increasing order of spatial derivatives as higher-order corrections are attended. Both the resulting NSE's and, even more so, higher-order corrections grossly overestimate the actual high-k behavior. Moreover, the Burnett equations, obtained at the order following the NSE's, are unstable when subjected to short-wavelength perturbations which render these equations ill posed.

One is tempted to consider the hyperbolic 13-moment equations, of Grad, as the macroscopic analogy of the telegrapher's equation. However, Grad's equations are plagued by a number of mathematical difficulties that greatly reduce their viability, and in fact these equations have hardly ever been used to address real-world problems. On top of these difficulties and the complexity of Grad equations, one has to add that the mathematical sense in which these equations approximate the original process is not clear. The method of moments does not seem to be related in a natural way either to the Hilbert or to the Chapman-Enskog expansion.

The idea behind the *ad hoc* approach of Khonkin [2] is a better starting point. Though *per se* it is only partially correct, it can be made into a systematic asymptotic procedure in which *space and time are kept on equal footing.* The crucial step is to realize that it is necessary to balance time and space. To this end, in addition to the usual slow time scale, we have to introduce into the expansion a *fast time scale.* It turns out that the presence of these two time scales, in parallel with two spatial scales, captures the needed effects of memory. Note that the additional time scale has to be a fast one in order to balance the fast spatial scale. If two slow-time scales are employed, one obtains a direct version of the iterative Chapman-Enskog approach [3].

These ideas are not merely speculations. Recently they were applied to a simple kinetic problem—the Broadwell model [4]. The resulting hyperbolic equations reflect the symmetric role given to space and time. Their hyperbolic nature also assures freedom from paradoxes of infinite fluxes and infinite speed of propagation. Above all, we have verified that the ultraviolet part of these equations coincides to leading order with the corresponding part of the linearized Broadwell model [5]. At least, on the linearized level, the macroscopic equations reproduce the original spectrum at all wavelengths. Thus the derived macroscopic equations are valid far beyond the formal limits of the expansion.

On all accounts such an expansion is superior to the Chapman-Enskog procedure. (This is even more transparent if the expansion is carried further, beyond the level of the continuum.) Our expansion preserves the hyperbolic nature of the resulting equations with a modified stress variable. In contrast, in the Chapman-Enskog procedure the higher-order corrections enter as higher and higher-order gradients acting on the same variables. Moreover, certain effects included in our equations already at leading order will *never* be recovered in the Chapman-Enskog procedure. More details will be provided elsewhere [5].

However, some cautious remark is in order; our discussion assumed implicitly that the process is either linear or weakly nonlinear. In a strongly nonlinear process, in general, the spectral proximity of linearized spectra does not imply proximity of the resulting dynamics. For instance, under a large perturbation, nonlinear hyperbolic equations describing a macroscopic process may break down after a finite time, while the equations describing the microscopic process will continue to provide a perfectly acceptable solution.

We now turn to the technical aspects of this Rapid Communication. Following Goldstein [6] consider the voyage of a persistent random walker in τ intervals on an equally l-spaced one-dimensional (1D) lattice. Let $\sigma(x,t)$ be the probability density function associated with point x at time t; then $\sigma(x,t)$ satisfies the following equation:

$$\sigma(t+\tau,x)=\left[1-\frac{\tau}{2A}\right][\sigma(t,x-l)+\sigma(t,x+l)] - \left[1-\frac{\tau}{A}\right]\sigma(t-\tau,x) . \quad (1)$$

We assume that c, the correlation coefficient between the forward moving and reversing particle, is finite. This assures a finite mean speed when the limit $l=dx/2\searrow 0$, $\tau=dt\searrow 0$ is taken. The constant A is related to c via $c=1-dt/A$.

We now expand (1) in τ and l to express it as an operator equation

$$2[\cosh(\tau\partial_t)-1]\sigma+\frac{\tau}{A}[\cosh(l\partial_x)-e^{-\tau\partial_t}]\sigma =2[\cosh(l\partial_x)-1]\sigma . \quad (2)$$

A well-balanced continuum level is obtained, assuming that $V\equiv l/\tau$ remains finite in the limit. Then, either by taking the first terms in expansion or taking the limit, we obtain

$$\sigma_{tt}+\frac{1}{4A}\sigma_t=V^2\sigma_{xx} . \quad (3)$$

Equation (3) is known as the telegrapher's equation.

To continue, we take the Fourier transform of (2) in both space and time to obtain ($\partial_t\leftrightarrow i\omega$, $\partial_x\leftrightarrow ik$)

$$\sin^2\left[\frac{\omega\tau}{2}\right]-\frac{i\tau}{4A-2\tau}\sin\tau\omega=\sin^2\left[\frac{kl}{2}\right] . \quad (4)$$

Let $\omega=\omega_R+i\omega_I$; then (4) implies

$$\cos(\omega_R\tau)[\cosh(\omega_I\tau)-\tau\gamma\sinh(\omega_I\tau)]=\cos(kl) , \quad (5a)$$

$$\sin(\tau\omega_R)[\sinh(\tau\omega_I)-\tau\gamma\cosh(\omega_I\tau)]=0 , \quad (5b)$$

where $\gamma^{-1}=4A-2\tau$. For $\sin\tau\omega_R\neq 0$, Eq. (5) leads to

$$\frac{\cos(\tau\omega_R)}{\cosh(\tau\omega_I)}=\cos(lk) \quad (6a)$$

and

$$\tau\gamma=\tanh(\tau\omega_I) . \quad (6b)$$

Using (6b) in (6a) yields

$$\cos(\tau\omega_R)=\frac{\cos(lk)}{\sqrt{1-\tau^2\gamma^2}} . \quad (7)$$

Equation (7) describes a discrete process and as such the shortest wavelength is given by the size of the spatiotemporal lattice. To compare it with a continuum analog, we envision a lattice that repeatedly is made dense with the intention of ultimately proceeding to the limit. If the problem is posed as an integral operator on a continuum, one can carry on high-**k** expansion of the discrete process and take such a limit.

A slightly more transparent form of (7) is obtained after the expansion of the radical

$$\sin^2\left[\frac{\tau\omega_R}{2}\right]=-\frac{\tau^2\gamma^2}{4}+\left[1+\frac{\tau^2\gamma^2}{2}\right]\sin^2\left[\frac{\tau kV}{2}\right] . \quad (8)$$

A glance at the dispersion relation (8) reveals a gap zone in the $\tau\gamma\ll 1$ vicinity of $kl=2\pi n$. In these gaps waves are attenuated and cannot propagate. This result is to be compared with the spectrum of its low-k descendent, the TE, which yields

$$\omega_R^2+\frac{\gamma^2}{4}=k^2V^2 . \quad (9)$$

For $k^2\le\gamma^2/V^2$ the behavior is purely diffusive ($\omega_R=0$) with attenuation given as $2\omega_I=\gamma\pm\sqrt{\gamma^2-4k^2V^2}$. Disregarding the fact that (9) was derived for small k's, we look at its behavior for arbitrary k and observe that the spectra of (8) and (9) overlap very nicely *everywhere* but at the higher gap zones, which are missed by the telegrapher's equation. But even there, the absolute difference between the approximation and its antecedent is small, confirming our assertion on the proximity of the spectra. Note that V, the common characteristic speed of all wavelengths, was preserved by the small-k expansion.

The spectral proximity should not be taken for granted. A far more typical example is that of the totally time-continuous uncorrelated random walker, wandering according to

$$\sigma_t = \frac{D_0}{l^2}[\sigma(t,x+l) - 2\sigma(t,x) + \sigma(t,x-l)] \tag{10}$$

and the Fokker-Planck limit ($l \searrow 0$,)

$$\sigma_t = D_0 \sigma_{xx} \ . \tag{11}$$

In Fourier space, (11) and (10) are

$$\omega_I = -D_0 k^2 \tag{12a}$$

and

$$\omega_I = -D_0 \frac{\sin^2(kl/2)}{l^2} \ , \tag{12b}$$

respectively. Written as an infinite sum, the right-hand side (rhs) of (12b) is essentially the Kramers-Moyal expansion. The rhs of (12a) is the first term in this expansion. Clearly, *for large* k the spectra of (12a) and (12b) have nothing in common. A similar phenomenon occurs if one considers the time-discrete uncorrelated random walker. These examples describe a typical state of affairs wherein the continuum model is neither expected nor capable of describing properly the short-wavelength behavior of its antecedent. Moreover, it follows from Pawula's theorem [7,8] that a higher-order polynomial approximation of (12b) will not eliminate the short-wavelength difficulty. For instance, extending (12a) to the next level of expansion, (12b) yields

$$\omega_I = -D_0 \left[k^2 - \frac{l^2 k^4}{12} \right]$$

and leads to an ill-posed problem

$$\sigma_t = D_0 \left[\sigma_{xx} + \frac{l^2}{12} \sigma_{xxxx} \right] . \tag{13}$$

Taking an odd number of terms, $2n-1$, $n>1$, in the expansion of (12b) renders the problem well posed but will not preserve positivity. As we have recently demonstrated, to this end one has to replace (12b) with a *globally* bounded operator [7]. The Lorentzian $(1+l^2k^2/12)^{-1}$, being the simplest one, yields

$$\sigma_t = D_0 \sigma_{xx} + \frac{l^2}{12} \sigma_{xxt} \ . \tag{14}$$

Equation (4) is well posed and preserves positivity (it has a maximum principle). It also properly describes the response to an initial impulse. Equation (14) renders a much better approximation of (10) than provided by the Fokker-Planck equation (11), but not as good as the one given by the TE.

We further digress to note that the idea of regularization that was just applied to the Kramers-Moyal expansion has a quite wide range of applicability. It can be used to express the dynamics of dense anharmonic lattices [9]. We have used it to regularize the Chapman-Enskog expansion [10] and thus to overcome the ill-posed nature of the Burnett equations.

Though the TE is an excellent approximation of (1), nevertheless one may be interested to obtain a further approximation of the original persisting walk. However, for reasons which at this point are not completely clear, I was not able to obtain a meaningful expansion of (1) beyond the stage of the TE without ruining some of its crucial features. If the regularized expansion is asymmetric in k and ω, the crucial property of hyperbolicity is lost; if it is symmetric, then an additional, false characteristic emerges. To make things worse, the false characteristic is accompanied by an elliptic piece [e.g., $\omega^4 = k^4 \Longrightarrow (\omega^2 - k^2)(\omega^2 + k^2)$] making the resulting equation evolutionarily unstable.

Returning to our main concern we focus on the lesson to be learned from our simple problem. If on the level of the master equation the process has memory, its neglect for whatever reason(s) is perhaps *the most singular* perturbation of the process. Conversely, scaling the problem in a way that assures that the effects of memory are included provides us with a system that escapes the low-k trap.

On the basis of both the present problem and the fast-time expansion of the Broadwell model [5], we hope that an analogous expansion of the Boltzmann equation will yield well-behaved hyperbolic equations of gas dynamics. Such equations should provide a far more refined approximation of gas dynamics than it is possible to achieve with the Navier-Stokes equations.

But perhaps one can take this idea one step beyond the specific topic of kinetic theory of gases. Bringing the effects of memory into a field theory and carrying an expansion that preserves the basic invariants of the problem should result in a far better macroscopic description. At least, in the two examples reported here, the resulting hyperbolic continuum seems to provide a notable approximation of the original process.

Valuable comments were made in the final stage of this work by Dr. B. Hasslacher, Dr. G. Eyink, and Professor C. D. Levermore. This work was supported in part by AFOSR contract under Grant No. F49620-92-j-0054 and in part under Darpa-AFOSR Contract No. F49620-89-C-0087.

[1] H. R. Hoare, in *The Linear Gas,* edited by S. A. Rice, Advances in Chemical Physics Ser. 20 (Academic, New York, 1971), p. 135.

[2] A. D. Kohnkin, Fluid Mech. Sov. Res. **9**, 93 (1980).

[3] C. Cerignani, *The Boltzmann Equation and Its Applications* (Springer-Verlag, Berlin, 1988).

[4] J. E. Broadwell, Phys. Fluids **4**, 1243 (1964).

[5] P. Rosenau (unpublished).

[6] S. Goldstein, Q. J. Mech. Appl. Math. **2**, 129 (1951).

[7] C. R. Doering, P. S. Hagan, and P. Rosenau, Phys. Rev. A **36**, 985 (1987).

[8] R. F. Pawula, Phys. Rev. **162**, 186 (1967).

[9] P. Rosenau, Phys. Rev. B **36**, 5868 (1987).

[10] P. Rosenau, Phys. Rev. A **40**, 7193 (1989).